

# Bicovariograms and Euler characteristic II. Random fields excursions

Raphaël Lachieze-Rey

*Laboratoire MAP5, 45 Rue des Saints-Pères, 75006 Paris*

*e-mail:* [raphael.lachieze-rey@parisdescartes.fr](mailto:raphael.lachieze-rey@parisdescartes.fr)

**Abstract:** Let  $f$  be a  $\mathcal{C}^1$  bivariate function with Lipschitz derivatives, and  $F = \{x : f(x) \leq \lambda\}$  a level set of  $f$ , with  $\lambda \in \mathbb{R}$ . We give a new expression of the Euler characteristic of  $F$  in terms of the three-points indicator functions of the set. If  $f$  is a two-dimensional  $\mathcal{C}^1$  random field and the derivatives of  $F$  have Lipschitz constants with finite moments of sufficiently high order, this new approach yields an expression of the mean Euler characteristic in terms of the field third order marginal. We provide sufficient conditions and explicit formulas for Gaussian fields, relaxing the usual  $\mathcal{C}^2$  Morse hypothesis.

**MSC 2010 subject classifications:** Primary 60G60, 60G15; secondary 28A75, 60G10, 60D05, 52A22.

**Keywords and phrases:** Euler characteristic, Random fields, Gaussian processes, covariograms, intrinsic volumes,  $\mathcal{C}^{1,1}$  functions .

## 1. Introduction

The geometry of random fields excursion sets has been a subject of intense research over the last two decades. Many authors are concerned with the computation of the mean [3, 4, 5, 6, 8, 12, 13] or variance [14, 22] of the Euler characteristic, denoted by  $\chi$  here.

As an integer-valued quantity, the Euler characteristic can be easily measured and used in many estimation and modelisation procedures. It is an important indicator of the porosity of a random media [7, 17, 26], it is used in brain imagery [19, 27], astronomy, [22, 23, 25], and many other disciplines. See also [2] for a general review of applied algebraic topology.

Most of the available works on random fields use the results gathered in the celebrated monograph [6], or similar variants. In this case, theoretical computations of the Euler characteristic emanate from Morse theory, where the focus is on the local extrema of the underlying field [8], instead of the set itself. For the theory to be applicable, the functions must be  $\mathcal{C}^2$  and satisfy the Morse hypotheses, which conveys some restrictions on the set itself.

The expected Euler characteristic also turned out to be a widely used approximation of the distribution function of the maximum of a Morse random field, and attracted much interest in this direction, see [3, 9, 27]. Indeed, for large  $r > 0$ , a well-behaved field rarely exceeds  $r$ , and if it does, it is likely to have

a single highest peak, which yields that the level set of  $f$  at level  $r$ , when not empty, is most often simply connected, and has Euler characteristic 1. In this fashion,  $\mathbf{E}\chi(\{f \geq r\}) \approx \mathbf{P}(\sup f \geq r)$ , which provides an additional motivation to compute the mean Euler characteristic of random fields.

Even though [4] provides an asymptotic expression for some classes of infinitely divisible fields, most of the tractable formulae concern Gaussian fields. One of the ambition of this paper is to provide a formula that is tractable in a rather general setting, and also works in the Gaussian realm.

### Approach and main results

Given a set  $A \subset \mathbb{R}^2$ , let  $\Gamma(A)$  be the class of its bounded connected components. We say that a set  $A$  is *admissible* if  $\Gamma(A)$  and  $\Gamma(A^c)$  are finite, and in this case its Euler characteristic is defined by

$$\chi(A) = \#\Gamma(A) - \#\Gamma(A^c),$$

where in all the paper  $\#$  denotes the cardinality of a set. The theoretical results of Adler and Taylor [6] regarding random excursions require second order differentiability of the underlying field  $f$ , but the expression of the mean Euler characteristic only involves the first-order derivatives, suggesting that second order derivatives do not matter in the computation of the Euler characteristic. In the words of Adler and Taylor (Section 11.7), regarding Formula (11.7.6), it is a *rather surprising fact that the [mean Euler characteristic of a Gaussian field] depends on the covariance of  $f$  only through some of its derivatives at zero, that is, only through the variance and second-order spectral moments. This is particularly surprising in view of the fact that the definition of the  $\mu_k$  depends quite strongly on the  $f_{i,j}$* , where the  $\mu_k$  are topological indexes depending on the Morse structure of  $f$ . We present here a new method, based on tools from image analysis, for which the second order differentiability is not needed. The results are valid for  $\mathcal{C}^1$  fields with Lipschitz derivatives, also called  $\mathcal{C}^{1,1}$  fields, relaxing slightly the classical  $\mathcal{C}^2$  Morse hypothesis.

Our main result relies on a preliminary work [20] connecting smooth sets Euler characteristic and variographic tools. For  $m, q \geq 1$ , points  $x_1, \dots, x_m, y_1, \dots, y_q$  of  $\mathbb{R}^2$  and  $A \subset \mathbb{R}^2$ , note the corresponding *polyvariogram*

$$\delta_{x_1, \dots, x_m}^{y_1, \dots, y_q}(F) = \text{Vol}((F + x_1) \cap \dots \cap (F + x_m) \cap (F + y_1)^c \cap \dots \cap (F + y_q)^c).$$

The function  $x \mapsto \delta_{0,x}(F)$  is called *covariogram*, and the function  $(x, y) \mapsto \delta_{0,x,y}$  the *bicovariogram* of  $F$ . See [21, Chap. 3.1] for more insights on variographic tools.

Note by  $\partial A$  the topological boundary of a set  $A \subset \mathbb{R}^2$ . We call here *regular set* a compact set  $F$  such that  $\partial F$  is a  $\mathcal{C}^1$  submanifold of  $\mathbb{R}^2$  and the mapping associating a point  $x \in \partial F$  to the outward normal vector of  $F$  in  $x$ ,  $\mathbf{n}_F(x)$ , is Lipschitz. Call  $\mathbf{u}_1, \mathbf{u}_2$  the canonical unit vectors of  $\mathbb{R}^2$ . It is proved in [20, Th. 1.6] that under these conditions, for  $\varepsilon$  small enough,

$$\chi(F) = \varepsilon^{-2}(\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F) - \delta_{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}^0(F)).$$

This expression could be equivalently expressed directly in terms of bicovariograms, using the relation  $\delta_0^{x,y} = \delta_0 - \delta_{0,x} - \delta_{0,y} + \delta_{0,x,y}$ ,  $x, y \in \mathbb{R}^2$ .

If  $F = \{x : f(x) \leq \lambda\}$  for some  $\lambda \in \mathbb{R}$  and a sufficiently smooth bi-variate function  $f$ , the formula above becomes

$$\chi(F) = \varepsilon^{-2} \int_{\mathbb{R}^2} \left[ \delta^\varepsilon(x, f, \lambda) - \delta^{-\varepsilon}(x, -f, -\lambda) \right] dx, \quad (1)$$

where

$$\delta^\eta(x, f, \lambda) = \mathbf{1}_{\{f(x) \leq \lambda, f(x+\eta \mathbf{u}_1) > \lambda, f(x+\eta \mathbf{u}_2) > \lambda\}}, \eta \in \mathbb{R}.$$

In the context of a random field  $f$ , this approach seems new in the literature. We see that, under suitable conditions, one can compute the mean Euler characteristic of random level sets solely in terms of the values taken by the third order marginal of the field in triples of arbitrarily close arguments, when classical methods usually require to know the distribution of the Hessian matrix of the field at each point. Let us write a simplified form of our main result here. A more general statement can be found in section 3.

**Theorem 1.** *Let  $W = [0, a] \times [0, b]$ ,  $a, b > 0$ . Let  $f$  be a  $C^1$  real random field defined on a neighbourhood of  $W$  with locally Lipschitz partial derivatives  $\partial_1 f, \partial_2 f$ ,  $\lambda \in \mathbb{R}$ , and assume  $F := \{f \leq \lambda\} \subset W$ . Assume furthermore that the following conditions are satisfied:*

- (i) *For some  $\kappa > 0$ , for  $x \in W$ , the density of the random vector  $(f(x), \partial_1 f(x), \partial_2 f(x))$  is bounded by  $\kappa$  on  $\mathbb{R}^3$ .*
- (ii) *There is  $p > 3$ ,  $\alpha \in (2/p, 1 - 1/p)$ ,  $\eta > 0$  such that for  $i = 1, 2$ ,*

$$\mathbf{E}[\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_i f)^{p(2-\alpha)+\eta}] < \infty.$$

*Then  $\chi(F)$  is a well-defined integrable random variable and*

$$\mathbf{E}\chi(F) = \lim_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon \mathbb{Z}^2} \mathbf{E}\delta^\varepsilon(x, f, \lambda) - \mathbf{E}\delta^{-\varepsilon}(x, -f, -\lambda) \quad (2)$$

$$= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{\mathbb{R}^2} \left[ \mathbf{E}\delta^\varepsilon(x, f, \lambda) - \mathbf{E}\delta^{-\varepsilon}(x, -f, -\lambda) \right] dx. \quad (3)$$

Theorem 9 actually features a result where  $f$  is defined on the whole plane and the level sets of  $F$  are observed through a bounded window  $W$ , as is typically the case for level sets of non-trivial stationary fields, but the intersection with  $\partial W$  requires additional notation and care. See Theorem 10 for a result tailored to deal with stationary fields excursions. Also, the density hypothesis (i) can be relaxed, in which case one has to study directly the *entanglement points* of  $f$ 's level sets, see (7).

Theorem 12 is specialised to the case where  $f$  is a Gaussian field. Under the additional hypotheses that  $f$  is stationary and isotropic, we retrieve in Theorem 13 the standard results of [6], assuming only  $\mathcal{C}^{1,1}$  regularity.

## 2. $\mathcal{C}^{1,1}$ functions

Let  $f$  be a function of class  $\mathcal{C}^1$  over some domain  $W \subset \mathbb{R}^2$ , and  $\lambda \in \mathbb{R}$ . Define

$$F := F_\lambda(f) = \{x \in W : f(x) \leq \lambda\}, \quad F_{\lambda-}(f) = \{x \in W : f(x) < \lambda\}.$$

Remark that  $F_{\lambda-}(f) = (F_{-\lambda}(-f))^c$ . If we assume that  $\nabla f$  does not vanish on  $\partial F_\lambda(f)$ ,  $\partial F_\lambda(f) = \partial F_{\lambda-}(f) = f^{-1}(\{\lambda\})$ , and this set is furthermore negligible, as a 1-dimensional manifold.

According to [15, 4.20],  $\partial F_\lambda(f)$  is regular, in the sense given in the introduction, if and only if  $\nabla f$  is locally Lipschitz and does not vanish on  $\partial F_\lambda(f)$ . This condition is necessary to prevent  $F$  from having infinitely many connected components, which would make Euler characteristic not properly defined, see [20, Remark 1.15]. We call  $\mathcal{C}^{1,1}$  function a differentiable function which gradient is a locally Lipschitz mapping. Those functions have been mainly used in optimisation problems, and as solutions of some PDEs, see for instance [18]. They can also be characterised as the functions which are locally *semiconvex* and *semiconcave*, see [11].

The result presented below shows that the Lipschitzness of  $\nabla f$  is also sufficient for the approximation (1) to hold. It seems therefore that the assumptions of Theorem 2, below, are the minimal ones ensuring the Euler characteristic to be computable with the covariogram method.

### Observation window

An aim of the present paper is to advocate the power of variographic tools for computing the Euler characteristic of random fields excursions. Since many applications are concerned with stationary random fields on the whole plane, we have to study the intersection of excursions with bounded windows, and assess the quality of the approximation.

To this end, call admissible rectangle of  $\mathbb{R}^2$  any set  $W = I \times J$  where  $I$  and  $J$  are closed (and possibly infinite) intervals of  $\mathbb{R}$ , and note  $\text{corners}(W)$  its corners, which number is between 0 and 4. Then call *polyrectangle* a finite union  $W = \cup_i W_i$  where each  $W_i$  is an admissible rectangle, and for  $i \neq j$ ,  $\text{corners}(W_i) \cap \text{corners}(W_j) = \emptyset$ . Call  $\mathcal{W}$  the class of polyrectangles.

We note by  $\text{corners}(W)$  the set of apparent corners of  $W$ , and for  $x \in \partial W \setminus \text{corners}(W)$ , note  $\mathbf{n}_W(x)$  the outward normal unit vector of  $W$  in  $x$ . We also call *edge of  $W$*  a maximal segment of  $\partial W$ , i.e. a segment  $[x, y] \subset \partial W$  that is not strictly contained in another such segment of  $\partial W$  (in this case,  $x, y \in \text{corners}(W)$ ).

Given a measurable set  $A \subset \mathbb{R}^2$  and  $\varepsilon > 0$ , we introduced in the companion paper the Gauss approximation  $A^\varepsilon$  of  $A$ , and formula (1) is the result of the fact that for  $\varepsilon$  sufficiently small,  $\chi((F_\lambda(f) + x)^\varepsilon) = \chi(F_\lambda(f))$  for  $x \in \mathbb{R}^2$  for a suitable function  $f$ . Even though it does not matter in the present paper, we

recall that this approximation is defined by

$$A^\varepsilon = \bigcup_{x \in \varepsilon \mathbb{Z}^2 \cap A} (x + \varepsilon[-1/2, 1/2]^2), \quad A \subset \mathbb{R}^2.$$

**Theorem 2.** *Let  $f$  be a  $\mathcal{C}^{1,1}$  function of  $\mathbb{R}^2$ , and  $W \in \mathcal{W}$ . Let  $\lambda \in \mathbb{R}$  such that*

1.  $F := F_\lambda(f) \cap W$  is bounded
2. For  $x \in W$  such that  $f(x) = \lambda$ , we have  $\nabla f(x) \neq 0$
3. For  $x \in \text{corners}(W)$ ,  $f(x) \neq \lambda$ .
4. For  $x \in \partial W$  such that  $f(x) = \lambda$ ,  $\mathbf{n}_F(x)$  is not colinear with  $\mathbf{n}_W(x)$ .

Then for  $\varepsilon$  sufficiently small

$$\begin{aligned} \chi(F) &= \chi(F^\varepsilon) \\ &= \sum_{x \in \varepsilon \mathbb{Z}^2} \delta^\varepsilon(x, f, \lambda) - \delta^{-\varepsilon}(x, -f, -\lambda) \\ &= \varepsilon^{-2} \int_{\mathbb{R}^2} [\delta^\varepsilon(x, f, \lambda) - \delta^{-\varepsilon}(x, -f, -\lambda)] dx \\ &= \varepsilon^{-2} [\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F) - \delta_{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}^0(F)]. \end{aligned} \tag{4}$$

The proof is a direct application of [20, Theorem 1.7] to  $F$ , using also the fact that for a.e.  $x \in \mathbb{R}^2$ ,  $\delta^\varepsilon(x, f, \lambda) = \mathbf{1}_{\{f(x) > \lambda, f(x + \varepsilon \mathbf{u}_1) \leq \lambda, f(x + \varepsilon \mathbf{u}_2) \leq \lambda\}}$ .

**Remark 3.** See [20, Remark 1.8] to track how small  $\varepsilon$  should be chosen, in function of the geometry of  $F$  and  $W$ .

### 2.1. Topological estimates

The next result is very general and does not concern directly the Euler characteristic. Its purpose is to bound the number of connected components of  $(F_\lambda(f) \cap W)^\varepsilon$  in view of applying Lebesgue's theorem to the point-wise convergence (4). It also provides a general bound for controlling the number of connected components of the excursion of a regular function, independently of the grid mesh.

Theorem 1.16 in the companion paper [20] features a bound on  $\chi(F \cap W)$  in terms of the number of occurrences of local configurations called *entanglement points of  $F$* , formally introduced through the notations  $\mathcal{N}_\varepsilon(F)$  and  $\mathcal{N}'_\varepsilon(F, W)$ . Roughly, an entanglement point occurs when two close points of  $F$  are connected by a tight path in  $F$ . As a consequence, if  $F$  is sampled with an insufficiently high resolution in this region, the connecting path is not detected, and  $F$  looks locally disconnected. The formal definitions of these quantities are recalled in due course along the proof of the following result. Denote by  $\text{Lip}(g)$  the Lipschitz constant of a Lipschitz function  $g$  going from a metric space to another.

**Theorem 4.** *Let  $W \in \mathcal{W}$ , and  $f : W \rightarrow \mathbb{R}$  be a  $\mathcal{C}^{1,1}$  function. Let  $F = F_\lambda(f)$  or  $F = F_{\lambda-}(f)$  for some  $\lambda \in \mathbb{R}$ . Assume that  $\nabla f$  does not vanish on  $\partial F$  and*

$F \cap W$  is bounded. Let  $\alpha, \beta, \alpha', \beta' > 0$  such that  $\alpha + 2\beta > 2, \alpha' + \beta' > 1$ . Let

$$I_{\alpha, \beta} := \text{Lip}(f)^\alpha (\text{Lip}(\partial_1 f)^{2\beta} + \text{Lip}(\partial_2 f)^{2\beta}) \int_W \frac{1}{|f(x) - \lambda|^\alpha |\partial_1 f(x) \partial_2 f(x)|^\beta} dx$$

$$I'_{\alpha', \beta'} := \text{Lip}(f)^{\alpha'} \sum_{i=1}^2 \left[ \text{Lip}(\partial_i f)^{\beta'} \int_{\partial W} \frac{1}{|f(x) - \lambda|^{\alpha'} |\partial_i f(x)|^{\beta'}} \mathcal{H}^1(dx) \right]$$

where  $\mathcal{H}^1$  is the 1-dimensional Hausdorff measure. Then there is  $C > 0$  depending on  $\text{diam}(F \cap W)$  and some quantity  $\varepsilon(W) > 0$  such that for  $0 < \varepsilon < \varepsilon(W)$ ,

$$\max(\#\Gamma((F \cap W)^\varepsilon), \#\Gamma(F \cap W)) \leq C(I_{\alpha, \beta} + I'_{\alpha', \beta'} + \#\text{corners}(W)). \quad (5)$$

Let us give a few remarks before the proof.

**Remark 5.** In the case where  $W = \mathbb{R}^2$ , only the first term remains in the right hand side of (5).

**Remark 6.** Similar results hold if  $f$ 's partial derivatives are only assumed to be Hölder, i.e. if there is  $\delta > 0$  and  $H_i > 0, i = 1, 2$  such that  $\|\partial_i f(x) - \partial_i f(y)\| \leq H_i \|x - y\|^\delta$  for  $x, y$  such that  $[x, y] \subset W$ . We don't treat such cases here because, as noted at the beginning of Section 2, if the partial derivatives are not Lipschitz, the level set is not regular enough to compute the Euler characteristic from the bicovariogram.

**Remark 7.** Calling  $B_\varepsilon$  the right hand term of (5) and noticing that  $F_{\lambda-}(f)^c$  is a level set of  $-f$ , an easy reasoning yields (see [20, Remark 1.17])

$$|\chi((F_\lambda(f) \cap W)^\varepsilon)| \leq 2B_\varepsilon.$$

**Notation** We need some general notation before turning to the proof. Denote by  $\mathcal{S}^1$  the set of unit vectors in  $\mathbb{R}^2$ . For a set  $A \subset \mathbb{R}^2$ , and  $r > 0$ , call  $A^{\oplus r} = \{x : d(x, A) \leq r\}$ , where  $d(x, A)$  is the Euclidean distance between  $x$  and  $A$ . We also note, for  $x \in \mathbb{R}^2$ ,  $(x_{[1]}, x_{[2]})$  its coordinates in the canonical basis. If  $\varphi$  is an application with values in  $\mathbb{R}^2$ , denote its coordinates by  $(\varphi(\cdot)_{[1]}, \varphi(\cdot)_{[2]})$ .

We also denote by  $\text{cl}(A), \text{int}(A)$  the topological closure and interior of a set  $A \subset \mathbb{R}^2$ .

At some point in the proof, and later in the article, we need the following simple but very convenient notation: For  $i \in \{1, 2\}$ , write

$$i' = \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{if } i = 2. \end{cases}$$

*Proof of Theorem 4.* We make only the proof for  $\lambda = 0$  for notational simplification. Let us first bound  $\#\Gamma(F \cap W)$ . Call in this proof  $\Gamma := \{C : C \in \Gamma(F \cap W), C \subseteq W\}$  and  $\Gamma' := \Gamma(F \cap \partial W)$ , so that  $\#\Gamma(F \cap W) \leq \#\Gamma + \#\Gamma'$ .

Let  $C \in \Gamma$ . Let  $x_C \in \text{cl}(C)$  such that  $f(x_C) = \inf_C f$ . Since  $\partial C \subset \partial F$  and  $\nabla f \neq 0$  on  $\partial F$ ,  $x_C \in \text{int}(C)$ , whence  $\nabla f(x_C) = 0$ . Denote by  $B[x, r]$  the

ball centred in  $x$  with radius  $r$  for the  $\infty$ -norm on  $\mathbb{R}^2$ . Call  $r_C = \sup\{r \geq 0 : B[x_C, r] \subseteq C\} > 0$ , and let  $B_C = B[x_C, r_C] \subset F \cap W$ . It is clear that for  $C, C' \in \Gamma$  distinct,  $C \cap C' = \emptyset$ . Call for  $n \in \mathbb{Z}$

$$\Gamma_n = \{C \in \Gamma : r_C \in [2^{-n}, 2^{-n+1})\},$$

Since  $F$  is bounded, there is  $n_0 = n_0(\text{diam}(F \cap W))$  such that  $\Gamma_n = \emptyset$  for  $n < n_0$ .

Let  $n \in \mathbb{Z}$ . For each  $C \in \Gamma_n$ , there is  $x \in (2^{-n}\mathbb{Z}^2 \cap B_C) \subset W$  and this is the only  $B_C$  to which  $x$  belongs. Also, since  $\partial F$  hits  $B[x_C, r_C]$ , it also hits  $B[x_C, 2^{-n+1}]$ , and for  $x \in B[x_C, 2^{-n+1}]$ ,  $\partial F \cap B[x, 2^{-n+2}] \neq \emptyset$ . Then

$$\begin{aligned} \#\Gamma &= \sum_{n \geq n_0} \#\Gamma_n \\ &\leq \sum_{n \geq n_0} \sum_{x \in 2^{-n}\mathbb{Z}^2 \cap W} \mathbf{1}_{\{x \in B_C \text{ for some } C \in \Gamma_n\}} \\ &\leq \sum_{n \geq n_0} \sum_{x \in 2^{-n}\mathbb{Z}^2 \cap W} \mathbf{1}_{\{\text{there is } x_C \in B[x, 2^{-n+1}] \cap W : \nabla f(x_C) = 0, \text{ and } \partial F \cap B[x, 2^{-n+2}] \neq \emptyset\}} \\ &\leq \sum_{n \geq n_0} \sum_{x \in 2^{-n}\mathbb{Z}^2 \cap W} \mathbf{1}_{\{f, \partial_1 f, \partial_2 f \text{ vanish in } B[x, 2^{-n+2}] \cap W\}} \\ &\leq \sum_{n \geq n_0} \sum_{x \in 2^{-n}\mathbb{Z}^2 \cap W} \frac{1}{\text{Vol}(B[x, 2^{-n}] \cap W)} \int_{B[x, 2^{-n}] \cap W} \mathbf{1}_{\{f, \partial_1 f, \partial_2 f \text{ vanish on } B[y, 2^{-n+3}] \cap W\}} dy. \end{aligned}$$

There is  $c = c(W) > 0$  such that for  $n \geq n_0$ ,  $\text{Vol}(B[x, 2^{-n}] \cap W) \geq 4^{-n}/c$ . Then

$$\begin{aligned} \#\Gamma &\leq c \sum_{n \geq n_0} 4^n \int_W \mathbf{1}_{\{|f(y)| \leq 2^{-n+4} \text{Lip}(f), |\partial_1 f(y)| \leq 2^{-n+4} \text{Lip}(\partial_1 f), |\partial_2 f(y)| \leq 2^{-n+4} \text{Lip}(\partial_2 f)\}} dy \\ &\leq c \sum_{n \geq n_0} 4^n \int_W \left| \frac{\text{Lip}(f) 2^{-n+4}}{f(y)} \right|^\alpha \left| \frac{\text{Lip}(\partial_1 f) 2^{-n+4}}{\partial_1 f(y)} \right|^\beta \left| \frac{\text{Lip}(\partial_2 f) 2^{-n+4}}{\partial_2 f(y)} \right|^\beta dy \\ &\leq c' \sum_{n \geq n_0} (2^{-n})^{-2+\alpha+2\beta} I_{\alpha, \beta}, \end{aligned} \tag{6}$$

where  $c'$  does not depend on  $\alpha, \beta$  (bounding  $\alpha$  and  $\beta$  by 1).

Let us now bound  $\#\Gamma'$  with a similar method, adapted to the dimension 1. Let  $C \in \Gamma'$  that does not touch  $\text{corners}(W)$ . Let  $i \in \{1, 2\}$  such that  $\mathbf{u}_i$  is orthogonal to  $\mathbf{n}_W(x), x \in C$ . The function  $f$  reaches an infimum on  $C$  in some point  $x_C \in C \setminus \partial F$ . Lagrange multipliers yields that  $\nabla f(x_C)$  is colinear with  $\mathbf{n}_W(x)$ , i.e. that  $\partial_{i'} f(x_C) = 0$ . Let  $r_C \geq 0$  maximal such that  $I_C := (x_C - r_C \mathbf{u}_i, x_C + r_C \mathbf{u}_i) \subseteq F \cap \partial W$ .

Let  $\Gamma'_n = \{C \in \Gamma' : C \cap \text{corners}(W) = \emptyset, r_C \in [2^{-n}, 2^{-n+1})\}$ . We have  $n'_0 = n'_0(W)$  such that  $\Gamma'_n = \emptyset$  for  $n < n'_0$ . For  $n \geq n'_0$ , let  $P_n$  be a  $2^{-n}$ -maximal packing of  $\partial W$ , i.e. a subset of  $\partial W$  such that every point of  $\partial W$  is within distance  $< 2^{-n}$  from  $P_n$ , but no two points of  $P_n$  are within distance

$2^{-n}$ . Standard results from Geometric Measure Theory yield that  $\#P_n \leq c2^n$  for some  $c > 0$ . For  $C \in \Gamma'_n$ ,  $I_C \cap P_n \neq \emptyset$ . Also, there is  $c' > 0$  such that  $\mathcal{H}^1(\partial W \cap B[x, 2^{-n}]) \geq 2^{-n}/c'$  for  $x \in \partial W$ ,  $n \geq 1$ . Then

$$\begin{aligned}
\#\Gamma' - \#\text{corners}(W) &\leq \sum_{n \geq n'_0} \#\Gamma'_n \\
&\leq \sum_{n \geq n'_0} \sum_{x \in P_n} \mathbf{1}_{\{x \in I_C \text{ for some } C \in \Gamma'_n\}} \\
&\leq \sum_{n \geq n'_0} \sum_{x \in P_n} \mathbf{1}_{\{\text{there is } i \in \{1,2\}, x_C \in B(x, 2^{-n}) \cap \partial W : \partial_i f(x_C) = 0; \text{there is } z \in B(x_C, 2^{-n+1}) \cap \partial W : f(z) = 0\}} \\
&\leq \sum_{n \geq n'_0} \sum_{x \in P_n} \mathbf{1}_{\{f, \partial_i f \text{ vanish on } B(x, 2^{-n+2}) \cap \partial W \text{ for some } i \in \{1,2\}\}} \\
&\leq \sum_{n \geq n'_0} \sum_{x \in P_n} \frac{1}{\mathcal{H}^1(\partial W \cap B(x, 2^{-n}))} \int_{\partial W \cap B(x, 2^{-n})} \mathbf{1}_{\{f, \partial_i f \text{ vanish on } B(y, 2^{-n+3}) \cap \partial W \text{ for some } i \in \{1,2\}\}} dy \\
&\leq \sum_{i=1}^2 \sum_{n \geq n'_0} c' 2^n \int_{\partial W} \mathbf{1}_{\{f, \partial_i f \text{ vanish on } B(y, 2^{-n+3})\}} \mathcal{H}^1(dy) \\
&\leq c'' \sum_{i=1}^2 \sum_{n \geq n'_0} (2^{-n})^{-1+\alpha+\beta} \int_{\partial W} \left| \frac{\text{Lip}(f)}{f(y)} \right|^\alpha \left| \frac{\text{Lip}(\partial_i f)}{\partial_i f(y)} \right|^\beta \mathcal{H}^1(dy),
\end{aligned}$$

with a technique similar to (6). Since only components touching  $\text{corners}(W)$  have not been accounted for in the previous computations, this yields the first part of the inequality,  $\#\Gamma(F \cap W) \leq C(I_{\alpha,\beta} + I'_{\alpha,\beta} + \#\text{corners}(W))$ .

We use [20, Theorem 1.16] to bound  $\#\Gamma((F \cap W)^\varepsilon)$ , for that we recall first the definition of  $\mathcal{N}_\varepsilon(F)$ ,  $\mathcal{N}'_\varepsilon(F, W)$ . For  $x, y \in \mathbb{R}^2$ , introduce  $P_{x,y}$  the closed square with side-length  $\varepsilon$  such that  $x$  and  $y$  are the midpoints of two opposite sides. Denote  $P'_{x,y} = \partial P_{x,y} \setminus \{x, y\}$ , which has two connected components. Then  $\{x, y\}$  is an *entanglement pair of points* of  $F$  if  $x, y \notin F$  and  $(P'_{x,y} \cup F) \cap P_{x,y}$  is connected. We call  $\mathcal{N}_\varepsilon(F)$  the family of such pairs of points. See Figure 1 for an example.

We introduce the notation  $\langle x, y \rangle = \varepsilon \mathbb{Z}^2 \cap [x, y] \setminus \{x, y\}$ , for  $x, y \in \varepsilon \mathbb{Z}^2$ . For the boundary version we also consider grid points  $x, y \in \varepsilon \mathbb{Z}^2 \cap W \cap F$ , on the same line or column of  $\varepsilon \mathbb{Z}^2$ , such that

- $x, y$  are within distance  $\varepsilon$  from one of the edges of  $W$  (the same edge for  $x$  and  $y$ )
- $\langle x, y \rangle \neq \emptyset$
- $\langle x, y \rangle \subseteq \varepsilon \mathbb{Z}^2 \cap F^c \cap F^{\oplus \varepsilon}$ .

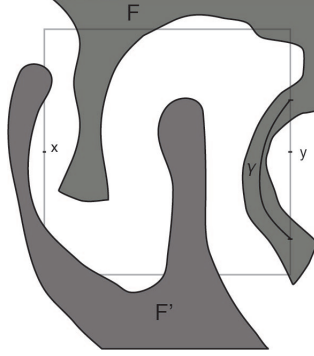
The family of such pairs of points  $\{x, y\}$  is noted  $\mathcal{N}'_\varepsilon(F; W)$ .

It is proved in [20, Theorem 1.16] that given any measurable set  $A$ ,

$$\#\Gamma((A \cap W)^\varepsilon) \leq 2\#\mathcal{N}_\varepsilon(A) \cap W^{\oplus \varepsilon} + 2\#\mathcal{N}'_\varepsilon(A, W) + \#\Gamma(A \cap W) + 2\#\text{corners}(W). \quad (7)$$



FIG 1. Entanglement point In this example,  $\{x, y\} \in \mathcal{N}_\varepsilon(F)$  because the two connected components of  $P'_{x,y}$ , in lighter grey, are connected through  $\gamma \subseteq (F \cap P_{x,y})$ . We don't have  $\{x, y\} \in \mathcal{N}_\varepsilon(F')$ .



It therefore only remains to bound  $\mathcal{N}_\varepsilon(F) \cap W^{\oplus \varepsilon}$  and  $\mathcal{N}'_\varepsilon(F, W)$  to achieve (5). For  $m \geq 1$  and a function  $g : A \subseteq \mathbb{R}^m \rightarrow \mathbb{R}, \varepsilon > 0$ , introduce the continuity modulus

$$\omega(g, A) = \sup_{x \neq y \in A} \|g(x) - g(y)\|.$$

We also note in the sequel

$$\text{Lip}(g, A) = \sup_{x \neq y \in A} \frac{|g(x) - g(y)|}{\|x - y\|}$$

if  $A$  is not the largest domain on which  $g$  is defined. The bound will follow from the following lemma.

**Lemma 8.** (i) For  $\{x, y\} \in \mathcal{N}_\varepsilon(F)$ , we have for some  $i \in \{1, 2\}$ ,

$$\begin{aligned} |f(x)| &\leq \omega(f, [x, y]) \leq \text{Lip}(f)\varepsilon \\ |\partial_i f(x)| &\leq \omega(\partial_i f, [x, y]) \leq \text{Lip}(\partial_i f)\varepsilon \\ |\partial_{i'} f(x)| &\leq 2\omega(\partial_i f, P_{x,y}) + \omega(\partial_{i'} f, P_{x,y}) \leq \sqrt{2}\varepsilon(2\text{Lip}(\partial_i f) + \text{Lip}(\partial_{i'} f)), \end{aligned}$$

and idem for  $y$ .

(ii) For  $x, y \in \mathcal{N}'_\varepsilon(F, W)$ , there is  $z = z(x, y) \in [x, y]$ ,  $i \in \{1, 2\}$ , such that

$$\begin{aligned} |f(z)| &\leq \text{Lip}(f)\varepsilon \\ |\partial_i f(z)| &\leq \text{Lip}(\partial_i f)\varepsilon. \end{aligned}$$

*Proof.* (i) Let  $x, y \in \mathcal{N}_\varepsilon(F)$ . The definition of  $\mathcal{N}_\varepsilon(F)$  yields a connected path  $\gamma \subseteq (F \cap P_{x,y})$  going through some  $z \in [x, y]$  and connecting the two connected components of  $P'_{x,y}$ . Since  $f(x) \geq 0$  and  $f(z) \leq 0$ , there is a point  $z'$  of  $[x, y]$  satisfying  $f(z') = 0$ . The mean value theorem yields the first point:  $|f(x)| \leq \omega(f, [x, y])$ . Note for later that for  $t \in P_{x,y}$   $|f(t)| \leq \omega(f, P_{x,y})$ .

We assume without loss of generality that  $[x, y]$  is horizontal. Let  $[z', z'']$  be the (also horizontal) connected component of  $F \cap [x, y]$  containing  $z$ . After choosing a direction on  $[x, y]$ ,  $z'$  and  $z''$  are entry and exit points for  $F$ , and their normal vectors  $\mathbf{n}_F(z'), \mathbf{n}_F(z'')$  point towards the outside of  $F$ . Therefore they satisfy  $\mathbf{n}_F(z')_{[1]} \mathbf{n}_F(z'')_{[1]} \leq 0$ , and so  $\partial_1 f(z') \partial_1 f(z'') \leq 0$ . This gives us by continuity the existence of a point  $w \in [x, y]$  such that  $0 = \partial_1 f(w)$ , whence  $|f_1(x)| \leq \omega(\partial_1 f, [x, y])$ . Note for later that  $|\partial_1 f(t)| \leq \omega(\partial_1 f, P_{x,y})$  on  $P_{x,y}$ . If  $[x, y]$  is vertical,  $\partial_2 f$  verifies the inequality instead. Let us keep assuming that  $[x, y]$  is horizontal for the sequel of the proof.

We claim that  $|\partial_2 f(x)| \leq 2\omega(\partial_1 f, P_{x,y}) + \omega(\partial_2 f, P_{x,y})$ , and consider two cases to prove it.

- First case  $\partial_2 f(z') \partial_2 f(z'') \leq 0$ , and by continuity we have  $w \in [x, y]$  such that  $0 = \partial_2 f(w)$ , whence  $|\partial_2 f(\cdot)| \leq \omega(\partial_2 f, P_{x,y})$  on the whole pixel  $P_{x,y}$ . The desired inequality follows.
- Second case  $\partial_2 f(z') > 0, \partial_2 f(z'') > 0$  (equivalent treatment if they are both  $< 0$ ). Assume for instance that  $z'$  is the leftmost point, and that  $|\partial_2 f(x)| > 2\omega(\partial_1 f, P_{x,y}) + \omega(\partial_2 f, P_{x,y})$ , otherwise the claim is proved. It implies in particular that  $|\partial_2 f(\cdot)| > 2\omega(\partial_1 f, P_{x,y})$  on the whole pixel  $P_{x,y}$ . For simplification purpose, and up to translating the whole problem, assume that the Euclidean coordinates of  $z'$  are  $(t_1, 0)$  and that of  $z''$  are  $(t_2, 0)$ , for some  $-\varepsilon/2 \leq t_1 \leq t_2 \leq \varepsilon/2$ . We can apply the implicit function theorem on  $I \subseteq [-\varepsilon/2, \varepsilon/2]$  to yield a  $\mathcal{C}^1$  function  $\varphi : I \rightarrow \mathbb{R}$  such that  $\varphi(t_1) = 0$  and  $f(t, \varphi(t)) = 0$  for  $t \in I$ . Differentiating in  $t$  yields

$$\varphi'(t) = -\frac{\partial_1 f(t, \varphi(t))}{\partial_2 f(t, \varphi(t))}.$$

We already proved that  $|\partial_1 f(t, \varphi(t))| < \omega(\partial_1 f, P_{x,y})$  as long as  $(t, \varphi(t)) \in P_{x,y}$ , and our hypotheses led us to  $|\partial_2 f(\cdot, \varphi(\cdot))| > 2\omega(\partial_1 f, P_{x,y})$ , whence  $|\varphi'(t)| < \frac{1}{2}$  as long as  $t \in I, |\varphi(t)| \leq \varepsilon/2$ . It implies that we can choose  $I = [-\varepsilon/2, \varepsilon/2]$ , and still have  $|\varphi'(t)| < 1/2, |\varphi(t)| < \varepsilon/2$  on  $I$ . Reasoning similarly around  $z''$ , there is  $\psi$  smooth defined on  $I$  such that  $|\psi(t)| < \frac{1}{2}\varepsilon$ . The functions  $\varphi$  and  $\psi$  never reach the upper boundary of the pixel. Give the explicit names  $P_{x,y}^{up}$  and  $P_{x,y}^{down}$  to the two open halves of the pixel, and remark that due to the construction of  $z', z''$  and  $\varphi'(t_1) > 0, \psi'(t_2) < 0$  the graphs of  $\varphi$  and  $\psi$  on  $(t_1, t_2)$  are contained in  $P_{x,y}^{up}$ . On resp. a left neighbourhood of  $t_1$  (resp. right neighbourhood of  $t_2$ ), the graph of  $\varphi$  (resp.  $\psi$ ) is contained in  $P_{x,y}^{down}$ . We have therefore  $\varphi, \psi \geq 0$  on  $(t_1, t_2)$  because  $[z', z''] \subseteq F$ . Since also  $\varphi(t_2) < \varepsilon/2, \psi(t_1) < \varepsilon/2$ , the two graphs necessarily meet on  $(t_1, t_2)$  at some value  $< \varepsilon/2$  (one can actually prove that  $\varphi = \psi$  on  $(t_1, t_2)$ , but it is not necessary for our purpose). It follows that the connected component  $C$  of  $F \cap P_{x,y}^{up}$  containing  $z, z', z''$  is a subset of

$$\{(t, r) \text{ with } t \in [t_1, t_2] \text{ and } r \leq \varphi(t)\} \subseteq [t_1, t_2] \times [0, \varepsilon/2).$$

In particular,  $C$  does not meet the upper component of  $\mathbf{P}'_{x,y}$ . At this point, up to changing  $z$ , we can assume that  $z$  is the last crossing point of  $\gamma$  with  $[x, y]$  before reaching the upper part of  $\mathbf{P}'_{x,y}$ , but then  $\gamma$  cannot be contained in  $C$ , which brings us to a contradiction.

We indeed proved that  $|\partial_2 f(x)| \leq 2\omega_1(\partial_2 f, \mathbf{P}_{x,y}) + \omega_2(\partial_2 f, \mathbf{P}_{x,y})$ .

(ii) Let now  $\{x, y\}$  be an element of  $\mathcal{N}'_\varepsilon(f, W)$ . We know that  $(x, y) \cap F^c \neq \emptyset$ . Let  $[z', z''] \subset [x, y]$  a connected component of  $F^c \cap [x, y]$ . If  $[z', z'']$  is, say, horizontal, since  $\mathbf{n}_F(\cdot)_{[1]}$  changes sign between  $z'$  and  $z''$ , so does  $\partial_1 f$ , and by continuity there is  $w \in [z', z'']$  where  $\partial_1 f(w) = 0$ . Calling  $z$  the closest point from  $w$  in  $(x, y)$ ,  $\|z - w\| \leq \varepsilon$ , and by definition of  $\mathcal{N}'_\varepsilon(F, W)$ ,  $z$  is also at distance  $\varepsilon$  from  $\partial F = \{f = 0\}$ . It follows that  $|\partial_1 f(z)| \leq \text{Lip}(\partial_1 f)\varepsilon$ ,  $|f(z)| \leq \text{Lip}(f)\varepsilon$ .  $\square$

To obtain the integral upper bounds from (5), note that given a point  $z \in W$ , there can be at most 12 distinct couples  $\{x, y\} \in \mathcal{N}_\varepsilon(F)$  such that  $z$  is within distance  $\varepsilon$  from  $\{x, y\}$ , and there is  $c > 0$  such that for every such  $x, y \in W$ ,  $\text{Vol}(B(x, \varepsilon) \cup B(y, \varepsilon) \cap W) \geq \varepsilon^2/c$ . It follows that, using Lemma 8,

$$\begin{aligned} \mathcal{N}_\varepsilon(F) &\leq \sum_{x, y \in \mathcal{N}_\varepsilon(F)} \sum_{i=1}^2 \mathbf{1}_{\{\text{for } z \in B(x, \varepsilon) \cup B(y, \varepsilon), |f(z)| \leq 2\text{Lip}(f)\varepsilon, |\partial_i f(z)| \leq 2\text{Lip}(\partial_i f)\varepsilon, |\partial_{i'} f(z)| \leq \sqrt{2}(\text{Lip}(\partial_1 f) + \text{Lip}(\partial_2 f))\varepsilon\}} \\ &\leq \sum_{i=1}^2 \sum_{x, y \in \mathcal{N}_\varepsilon(F)} \frac{c}{\varepsilon^2} \\ &\quad \int_{(B(x, \varepsilon) \cup B(y, \varepsilon)) \cap W} \mathbf{1}_{\{|f(z)| \leq 2\text{Lip}(f)\varepsilon, |\partial_i f(z)| \leq 2\text{Lip}(\partial_i f)\varepsilon, |\partial_{i'} f(z)| \leq \sqrt{2}(\text{Lip}(\partial_1 f) + \text{Lip}(\partial_2 f))\varepsilon\}} dz \\ &\leq 12c\varepsilon^{-2} \sum_{i=1}^2 \int_W \mathbf{1}_{\{|f(z)| \leq 2\text{Lip}(f)\varepsilon, |\partial_i f(z)| \leq 2\text{Lip}(\partial_i f)\varepsilon, |\partial_{i'} f(z)| \leq \sqrt{2}(\text{Lip}(\partial_1 f) + \text{Lip}(\partial_2 f))\varepsilon\}} dz, \end{aligned}$$

which gives  $\mathcal{N}_\varepsilon(F) \leq CI_{\alpha, \beta}$  with a technique similar to (6), using  $\alpha + 2\beta > 2$ , and  $\text{Lip}(\partial_1 f)^\beta \text{Lip}(\partial_2 f)^\beta \leq \text{Lip}(\partial_1 f)^{2\beta} + \text{Lip}(\partial_2 f)^{2\beta}$ .

Now, given  $w \in \partial W$ , there can be at most 3 pairs  $\{x, y\} \in \mathcal{N}'_\varepsilon(F)$  such that  $w$  is on the closest edge of  $W$  parallel to  $[x, y]$  and  $z = z(x, y)$  (defined in Lemma 8) is within distance  $3\varepsilon$  from  $w$ . We have  $\mathcal{H}^1(B(z, 3\varepsilon) \cap \partial W) \geq \varepsilon$ , because  $t$  is within distance  $2\varepsilon$  from  $\partial W$ . It follows that

$$\begin{aligned} \#\mathcal{N}'_\varepsilon(F, W) &\leq \sum_{x, y \in \mathcal{N}'_\varepsilon(F, W)} \sum_{i=1}^2 \mathbf{1}_{\{\text{for } w \in B(z, 3\varepsilon) \cap \partial W, |f(w)| \leq 4\text{Lip}(f)\varepsilon, |\partial_i f(w)| \leq 4\text{Lip}(\partial_i f)\varepsilon\}} \\ &\leq \sum_{i=1}^2 \sum_{x, y \in \mathcal{N}'_\varepsilon(F)} \frac{1}{\varepsilon} \int_{\partial W \cap B(z, 3\varepsilon)} \mathbf{1}_{\{|f(w)| \leq 4\text{Lip}(f)\varepsilon, |\partial_i f(w)| \leq 4\text{Lip}(\partial_i f)\varepsilon ; w \in B(z, 3\varepsilon) \cap \partial W\}} \mathcal{H}^1(dw) \\ &\leq \sum_{i=1}^2 \frac{3}{\varepsilon} \int_{\partial W} \mathbf{1}_{\{|f(w)| \leq 4\text{Lip}(f)\varepsilon, |\partial_i f(w)| \leq 4\text{Lip}(\partial_i f)\varepsilon\}} \mathcal{H}^1(dw), \end{aligned}$$

which gives  $\#\mathcal{N}'_\varepsilon(F, W) \leq CI'_{\alpha, \beta}$  with a technique similar to (6), using  $\alpha + \beta > 1$ .  $\square$

### 3. Mean Euler characteristic of random excursions

To avoid measurability issues, we call here  $\mathcal{C}^1$  random field over a set  $\Omega \subseteq \mathbb{R}^2$  the data of a family of random variables  $\{f(x); x \in \Omega\}$ , such that in each point  $x \in \Omega$ , the variables

$$\partial_i f(x) := \lim_{s \rightarrow 0} \frac{f(x + s\mathbf{u}_i)}{s}, i = 1, 2,$$

exist a.s., and the fields  $(\partial_i f(x), x \in \Omega), i = 1, 2$ , are a.s. continuous. See [1, 6] for a discussion on the regularity properties of random fields. Say that the random field is  $\mathcal{C}^{1,1}$  if the partial derivatives are a.s. Lipschitz.

Many sets of conditions allowing to take the expectation in (4) can be derived from Theorem 4, we give below a compromise between optimality and compactness. Given a random closed set  $F$ , call  $\text{supp}(F)$  the smallest compact set  $K$  satisfying  $F \subset K$  a.s. See [24] for a formal introduction to the theory of random closed sets.

**Theorem 9.** *Let  $f$  be a  $\mathcal{C}^{1,1}$  random field,  $\lambda \in \mathbb{R}$ , and  $F = \{f \leq \lambda\}$ . Let  $W \in \mathcal{W}$  such that  $\text{supp}(F) \cap W$  is bounded. Assume that the following conditions are satisfied:*

- (i) *For some  $\kappa > 0$ , for  $x \in W$ , the density of the random vector  $(f(x), \partial_1 f(x), \partial_2 f(x))$  is bounded by  $\kappa$  on  $\mathbb{R}^3$ .*
- (ii) *There is  $p > 3, \alpha \in (2/p, 1 - 1/p), \eta > 0$  such that for  $i = 1, 2$ ,*

$$\mathbf{E}[\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_i f)^{p(2-\alpha)+\eta}] < \infty.$$

*Then  $F \cap W \in \mathcal{A}$  a.s.,  $\mathbf{E}|\chi(F \cap W)| \leq \mathbf{E}\# \Gamma(F \cap W) + \mathbf{E}\# \Gamma((F \cap W)^c) < \infty$ , and we have*

$$\mathbf{E}\chi(F \cap W) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} (\mathbf{E}\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F \cap W) - \mathbf{E}\delta_{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}^0(F \cap W)),$$

*and the result can also be stated with a discrete sum instead of an integral, like in (2).*

Before giving the proof, we give an explicit expression, in the case where  $f$  is stationary, which is the adaptation of [20, Proposition 2.1]. Boundary terms will involve the perimeter of  $F$ , so we introduce the related notation below. Note  $\mathcal{C}_c^1$  the class of compactly supported functions on  $\mathbb{R}^2$ . For a measurable set  $A$ , and  $\mathbf{u} \in \mathcal{S}^1$ , define the variational perimeter of  $A$  in direction  $\mathbf{u}$  by

$$\text{Per}_{\mathbf{u}}(A) = \sup_{\varphi \in \mathcal{C}_c^1: \|\varphi(x)\| \leq 1} \int_A \langle \nabla \varphi(x), \mathbf{u} \rangle dx,$$

and the  $\|\cdot\|_\infty$ -perimeter

$$\text{Per}_\infty(A) = \text{Per}_{\mathbf{u}_1}(A) + \text{Per}_{\mathbf{u}_2}(A),$$

named like this because it is the analogue of the classical perimeter when the Euclidean norm is replaced by the  $\|\cdot\|_\infty$ -norm, see [16].

**Theorem 10.** *Let  $f$  be a  $\mathcal{C}^{1,1}$  stationary random field,  $\lambda \in \mathbb{R}$ , and  $W \in \mathcal{W}$  bounded. Assume that  $(f(0), \partial_1 f(0), \partial_2 f(0))$  has a bounded density, and that there is  $p > 3, \alpha \in (2/p, 1 - 1/p), \eta > 0$  such that for  $i = 1, 2$ ,*

$$\mathbf{E} \left[ \text{Lip}(f; W)^{\alpha p} \text{Lip}(\partial_i f; W)^{p(2-\alpha)+\eta} \right] < \infty.$$

Then the following limits exist:

$$\begin{aligned} \overline{\chi}(f; \lambda) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \left[ \mathbf{E} \delta^\varepsilon(0, f, \lambda) - \mathbf{E} \delta^{-\varepsilon}(0, -f, -\lambda) \right] \\ \overline{\text{Per}_{\mathbf{u}_i}}(f; \lambda) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \mathbf{P}(f(0) \leq \lambda, f(\varepsilon \mathbf{u}_i) > \lambda) \\ \overline{\text{Vol}}(f; \lambda) &:= \mathbf{P}(f(0) \leq \lambda), \end{aligned}$$

and we have, with  $\overline{\text{Per}_\infty} = \overline{\text{Per}_{\mathbf{u}_1}} + \overline{\text{Per}_{\mathbf{u}_2}}$ ,

$$\begin{aligned} \mathbf{E} \chi(F \cap W) &= \text{Vol}(W) \overline{\chi}(F) + \frac{1}{4} (\text{Per}_{\mathbf{u}_2}(W) \overline{\text{Per}_{\mathbf{u}_1}}(F) + \text{Per}_{\mathbf{u}_1}(W) \overline{\text{Per}_{\mathbf{u}_2}}(F)) \\ &\quad + \chi(W) \overline{\text{Vol}}(F) \end{aligned} \quad (8)$$

$$\mathbf{E} \text{Per}_\infty(F \cap W) = \text{Vol}(W) \overline{\text{Per}_\infty}(F) + \text{Per}_\infty(W) \overline{\text{Vol}}(F) \quad (9)$$

$$\mathbf{E} \text{Vol}(F \cap W) = \text{Vol}(W) \overline{\text{Vol}}(F). \quad (10)$$

The proof of Theorem 9, given below, establishes that the expectations contained in [20, (2.2)] are finite. Therefore the result above is a consequence of that proof and [20, Proposition 3.1].

**Remark 11.** This formula should give tractable results even for fields that are not Gaussian. A work in preparation will feature the application of this formula to infinitely divisible fields, following the papers of [4, 10] where asymptotic or one-dimensional formulae are given for some shot noise models.

*Proof of Theorem 9.* Assume without loss of generality  $\lambda = 0$ . We must first prove that  $F$  is a.s. locally regular. For  $W$  compact, let  $\psi$  be a function of class  $\mathcal{C}^\infty$  constant equal to 1 on  $W \cap \text{supp}(F)$  and vanishing on  $((W \cap \text{supp}(F))^{\oplus 1})^c$ . Then the level sets of  $f\psi$  are regular and coincide with the level sets of  $f$  on  $W$ , therefore the level sets of  $f$  are locally regular.

Let us prove that  $F$  satisfies a.s. the hypotheses of Theorem 2. Define

$$\begin{aligned} \Theta_1 &= \{x \in W : f(x) = 0, \nabla f(x) = 0\} \\ \Theta_{2,i} &= \{x \in \partial W : f(x) = 0, \partial_i f(x) = 0\}, i = 1, 2, \\ \Theta_3 &= \{x \in \text{corners}(W) : f(x) = 0\}. \end{aligned}$$

We will prove that  $\mathbf{E}\#\Theta = \mathbf{E}\#\Theta_{2,i} = \mathbf{E}\#\Theta_3 = 0$ , which yields that these sets are a.s. empty, and therefore that  $F$  satisfies a.s. the hypotheses of Theorem 2. We define  $\mathbf{P} = \varepsilon[-1/2, 1/2]^2$ . We have a.s.

$$\begin{aligned} \#\Theta_1 &= \lim_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon \mathbb{Z}^2} \mathbf{1}_{\{\Theta_1 \cap (x+\mathbf{P}) \neq \emptyset\}} \\ &\leq \lim_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap W^{\oplus \varepsilon}} \mathbf{1}_{\{d(x, f^{-1}(\{0\})) \leq \varepsilon, d(x, \partial_1 f^{-1}(\{0\})) \leq \varepsilon, d(x, \partial_2 f^{-1}(\{0\})) \leq \varepsilon\}} \\ &\leq \lim_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap W^{\oplus 1}} \mathbf{1}_{\{|f(x)| \leq \text{Lip}(f)\varepsilon, |\partial_1 f(x)| \leq \text{Lip}(\partial_1 f)\varepsilon, |\partial_2 f(x)| \leq \text{Lip}(\partial_2 f)\varepsilon\}}. \end{aligned}$$

Let  $\beta > 0$ . Fatou's lemma yields, with  $\alpha, p$  like in the theorem statement, and  $p' := 1 - 1/p$ ,

$$\begin{aligned} \mathbf{E}\#\Theta_1 &\leq \liminf_{\varepsilon \rightarrow 0} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap W^{\oplus \varepsilon}} \mathbf{E} \mathbf{1}_{\{|f(x)| \leq \text{Lip}(f)\varepsilon, |\partial_1 f(x)| \leq \text{Lip}(\partial_1 f)\varepsilon, |\partial_2 f(x)| \leq \text{Lip}(\partial_2 f)\varepsilon\}} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\alpha+2\beta} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap W^{\oplus \varepsilon}} \mathbf{E} \left[ \left( \frac{\text{Lip}(f)}{|f(x)|} \right)^\alpha \left( \frac{\text{Lip}(\partial_1 f)}{|\partial_1 f(x)|} \right)^\beta \left( \frac{\text{Lip}(\partial_2 f)}{|\partial_2 f(x)|} \right)^\beta \right] \\ &\leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\alpha+2\beta} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap W^{\oplus \varepsilon}} (\mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_1 f)^{\beta p} \text{Lip}(\partial_2 f)^{\beta p}])^{1/p} \\ &\quad (11) \\ &\quad \left( \mathbf{E} \left[ \frac{1}{|f(x)^{\alpha p'} \partial_1 f(x)^{\beta p'} \partial_2 f(x)^{\beta p'}|} \right] \right)^{1/p'}. \end{aligned}$$

We have similarly

$$\begin{aligned} \mathbf{E}\#\Theta_{2,i} &\leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\alpha+\beta} \sum_{x \in \varepsilon \mathbb{Z}^2 \cap \partial W^{\oplus \varepsilon}} \mathbf{E} \left[ \left( \frac{\text{Lip}(f)}{|f(x)|} \right)^\alpha \left( \frac{\text{Lip}(\partial_i f)}{|\partial_i f(x)|} \right)^\beta \right], i = 1, 2, \\ \mathbf{E}\#\Theta_3 &\leq \sum_{x \in \text{corners}(W)} \varepsilon^\alpha \mathbf{E} \left( \frac{\text{Lip}(f)}{|f(x)|} \right)^\alpha. \end{aligned}$$

Let  $\eta' > 0$  and assume  $\beta = 1 - \alpha/2 + \eta'$ . Then

$$\mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_1 f)^{\beta p} \text{Lip}(\partial_2 f)^{\beta p}] \leq \sqrt{\mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_1 f)^{2\beta p}] \mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_2 f)^{2\beta p}]}$$

and  $2\beta p = (2 - \alpha + 2\eta')p$ . Since  $\mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_1 f)^{(2-\alpha)p+\eta}] < \infty$ , and  $\eta'$  can be arbitrarily small,

$$\mathbf{E} [\text{Lip}(f)^{\alpha p} \text{Lip}(\partial_i f)^{2\beta p}] < \infty, i = 1, 2. \quad (12)$$

Then,  $\mathbf{E} \text{Lip}(f)^{\alpha p} \text{Lip}(\partial_i f)^{\beta p} < \infty, \mathbf{E} \text{Lip}(f)^{\alpha p} < \infty$ . To prove that the other expectation of (11) is also finite we need the following result. For any random

vector  $(U_1, U_2, U_3) \in \mathbb{R}^3$ , which density is bounded by  $\kappa$ , and  $\gamma_1, \gamma_2, \gamma_3 \geq 0$  we have

$$\begin{aligned} \mathbf{E} U_1^{-\gamma_1} U_2^{-\gamma_2} U_3^{-\gamma_3} &\leq \sum_{P \subseteq \{1,2,3\}} \mathbf{E} \left[ \mathbf{1}_{\{U_i \leq 1, i \in P; U_i > 1, i \notin P\}} \prod_{i \in P} U_i^{-\gamma_i} \right] \\ &\leq \sum_{P \subseteq \{1,2,3\}} \kappa \prod_{i \in P} \int_{[-1,1]} x^{-\gamma_i} dx \\ &\leq C(\kappa, \gamma_1, \gamma_2, \gamma_3). \end{aligned} \quad (13)$$

The latter quantity is finite if  $\gamma_i < 1, i = 1, 2, 3$ . If we assume in the display above that  $U_3$  is independent of  $(U_1, U_2)$ , then it yields that  $\mathbf{E} U_1^{-\gamma_1} U_2^{-\gamma_2}$  and  $\mathbf{E} U_3^{-\gamma_3}$  have a similar bound.

We have  $2/p < \alpha < 1 - 1/p$ , and  $1/p' = 1 - 1/p$ , whence  $\alpha < 1/p'$ , and  $1 - 1/p > 1 - \alpha/2$ , i.e.  $1/p' > \beta$  for  $\eta'$  sufficiently small. Therefore

$$\mathbf{E} \left[ \frac{1}{|f(x)^{\alpha p'} \partial_1 f(x)^{\beta p'} \partial_2 f(x)^{\beta p'}|} \right] \leq C(\kappa, \alpha p', \beta p', \beta p') < \infty.$$

It ultimately follows that for some  $C', C'' > 0$ ,

$$\mathbf{E} \# \Theta_1 \leq \liminf_{\varepsilon \rightarrow 0} C' \varepsilon^{2+\eta'} \# [W^{\oplus \varepsilon}]^\varepsilon \leq \liminf_{\varepsilon \rightarrow 0} C'' \varepsilon^{2\eta'} = 0.$$

The expectations of  $\Theta_{2,i}, \Theta_3$  are handled similarly, noting that  $\alpha + \beta > 1$ , whence a.s.  $F$  satisfies the hypotheses of Theorem 2.

Defining  $I_{\alpha,\beta}$  like in Theorem 4, we have

$$\begin{aligned} \mathbf{E} I_{\alpha,\beta} &\leq \mathbf{E} \left[ \text{Lip}(f)^\alpha (\text{Lip}(\partial_1 f)^{2\beta} + \text{Lip}(\partial_2 f)^{2\beta}) \int_W \frac{1}{|f(t)|^\alpha} \frac{1}{|\partial_1 f(t) \partial_2 f(t)|^\beta} dt \right] \\ &\leq [\mathbf{E} (\text{Lip}(f)^{p\alpha} (\text{Lip}(\partial_1 f)^{2\beta p} + \text{Lip}(\partial_2 f)^{2\beta p}))]^{1/p} \left[ \mathbf{E} \left( \int_W \frac{dt}{|f(t)|^\alpha |\partial_1 f(t) \partial_2 f(t)|^\beta} \right)^{p'} \right]^{1/p'} \\ &\leq [\mathbf{E} (\text{Lip}(f)^{p\alpha} (\text{Lip}(\partial_1 f)^{2\beta p} + \text{Lip}(\partial_2 f)^{2\beta p}))]^{1/p} \\ &\quad \left[ \mathbf{E} \left( \text{Vol}(W)^{p'/p} \int_W \frac{dt}{|f(t)|^{\alpha p'} |\partial_1 f(t) \partial_2 f(t)|^{\beta p'}} \right) \right]^{1/p'} \\ &\leq \text{Vol}(W)^{1/p} [\mathbf{E} (\text{Lip}(f)^{p\alpha} (\text{Lip}(\partial_1 f)^{2\beta p} + \text{Lip}(\partial_2 f)^{2\beta p}))]^{1/p} \\ &\quad \left( \int_W \mathbf{E} \left[ \frac{1}{|f(t)|^{\alpha p'} |\partial_1 f(t) \partial_2 f(t)|^{\beta p'}} \right] dt \right)^{1/p'} \\ &< \infty, \end{aligned}$$

using (12), (13). Similarly, taking  $\alpha' = \beta' = 1/2$ ,

$$\begin{aligned} \mathbf{E}I'_{\alpha',\beta'} &\leq \sum_{i=1}^2 \mathbf{E} \text{Lip}(f)^{1/2} \text{Lip}(\partial_i f)^{1/2} \int_{\partial W} \frac{1}{|f(t)|^{1/2} |\partial_i f(t)|^{1/2}} \mathcal{H}_1(dt) \\ &\leq \sum_{i=1}^2 \left( \mathbf{E} \text{Lip}(f)^{3/2} \text{Lip}(\partial_i f)^{3/2} \right)^{1/3} \text{Per}(W)^{1/3} \left( \int_{\partial W} \mathbf{E} \left[ \frac{1}{|f(t)|^{3/4} |\partial_i f(t)|^{3/4}} \right] \mathcal{H}^1(dt) \right)^{2/3} \\ &< \infty, \end{aligned}$$

whence Remark 7 yields,

$$\mathbf{E} \sup_{0 < \varepsilon < \varepsilon(W)} |\chi((F \cap W)^\varepsilon)| < \infty.$$

Therefore, applying Lebesgue's Theorem to the almost sure convergence (4) gives the result.  $\square$

### 3.1. Gaussian level sets

Let  $(f(x), x \in W)$  be a Gaussian field on some  $W \in \mathcal{W}$ . We assume throughout the section, mostly for notational simplification, that each variable  $f(x), x \in W$ , is centred and has variance 1. Let the covariance function be defined by

$$\sigma(x, y) = \mathbf{E}f(x)f(y), \quad x, y \in W.$$

The book [1] gives some background on Gaussian fields and their regularity. Theorem 2.2.2 states that if the derivative  $\partial^2 \sigma(x, y)/\partial x_{[i]} \partial y_{[i]}$  exists and is finite at each point  $(x, x), x \in W$ , the limits

$$\partial_i f(x) := \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon \mathbf{u}_i) - f(x)}{\varepsilon}, \quad i = 1, 2,$$

exist in the  $L^2$  sense, and they form a Gaussian field. Also, the covariance function of  $\partial_i f$  is  $(x, y) \mapsto \partial^2 \sigma(x, y)/\partial x_{[i]} \partial y_{[i]}, i = 1, 2$ .

We are interested here in the case where  $f$  is of class  $\mathcal{C}^{1,1}$ . For  $W$  bounded, since  $\|\partial_i f\| := \sup_{x \in W} |\partial_i f(x)|$  is a.s. finite,  $\|\partial_i f\|$  has finite moments of every order, see (2.1.4) in [6].

**Theorem 12.** *Let  $f$  be a  $\mathcal{C}^{1,1}$  Gaussian field on a bounded closed set  $W \in \mathcal{W}$ . Assume that for  $x \in W$ ,  $(f(x), \partial_1 f(x), \partial_2 f(x))$  is non-degenerate, and that for some  $\eta > 0$ , for  $i = 1, 2$ ,*

$$\mathbf{E} \text{Lip}(\partial_i f)^{4+\eta} < \infty.$$

*Then for any  $\lambda \in \mathbb{R}$ ,  $F = F_\lambda(f)$  satisfies the hypotheses of Theorem 9.*

*Proof.* Since  $\text{Lip}(f) = \sup_W \|\nabla f\|$  and  $\mathbf{E}\|\nabla f\|^q < \infty$  for any  $q \geq 1$ ,

$$\mathbf{E} \text{Lip}(f)^q < \infty.$$



Then, for some  $\eta' > 0$ , define  $p = 3 + \eta'$ ,  $\alpha = 1 - 1/p - \eta'$ ,  $q = 1/\eta'$ ,  $q' = 1/(1 - \eta')$ . For  $i \in \{1, 2\}$ ,

$$\begin{aligned} \mathbf{ELip}(f)^{p\alpha} \text{Lip}(\partial_i f)^{p(2-\alpha)+\eta'} &\leq (\mathbf{ELip}(f)^{p\alpha q})^{1/q} (\mathbf{ELip}(\partial_i f)^{(3+\eta')(1+1/(3+\eta')+\eta')/(1-\eta')})^{1-\eta'} \\ &\leq (\mathbf{ELip}(f)^{p\alpha q})^{1/q} (\mathbf{ELip}(\partial_i f)^{4+\eta''}) \end{aligned}$$

where  $\eta'' \leq \eta$  if  $\eta'$  is chosen small enough, whence indeed Theorem 9(ii) is satisfied.

Put for notational convenience  $f^{(0)} := f$ ,  $f^{(i)} = \partial_i f$ ,  $i = 1, 2$ . We have for  $i, j \in \{0, 1, 2\}$ ,

$$\begin{aligned} |\mathbf{E} f^{(i)}(x) f^{(j)}(x) - f^{(i)}(y) f^{(j)}(y)| \\ \leq \left| \mathbf{E} \left[ \left( f^{(i)}(x) - f^{(i)}(y) \right) f^{(j)}(x) \right] \right| + \left| \mathbf{E} \left[ f^{(i)}(y) \left( f^{(j)}(x) - f^{(j)}(y) \right) \right] \right| \\ \leq \mathbf{E} \sup_W |f^{(j)}| \text{Lip}(f^{(i)}) \|x - y\| + \mathbf{E} \sup_W |f^{(i)}| \text{Lip}(f^{(j)}) \|x - y\|, \end{aligned}$$

which yields that the covariance function with values in the space of  $3 \times 3$  matrices,

$$x \mapsto \Sigma(x) := \text{cov}(f(x), \partial_1 f(x), \partial_2 f(x))$$

is Lipschitz on  $W$ . In particular, since  $\det(\Sigma(x))$  does not vanish on  $W$ , it is bounded from below by some  $c > 0$ , whence the density of  $(f(x), \partial_1 f(x), \partial_2 f(x))$ ,  $x \in W$ , is uniformly bounded by  $\kappa := (2\pi)^{-3/2} c^{-1/2}$ , and assumption (i) from Theorem 9 is also satisfied.  $\square$

Let us give the mean Euler characteristic under the simplifying assumptions that the law of  $f$  is invariant under translations and rotations of  $\mathbb{R}^2$ . In combination with the constant variance assumption, it eases certain computations. This implies for instance that in every  $x \in \mathbb{R}^2$ ,  $f(x)$ ,  $\partial_1 f(x)$  and  $\partial_2 f(x)$  are independent, see for instance [6] Section 5.6 and (5.7.3). A nice feature of the following result is that the hypotheses on  $f$  match the result, in the sense that the mean Euler characteristic only depends on the properties of  $\nabla f$ , and that the number of times  $f$  should be continuously differentiable is only 1.

**Theorem 13.** *Let  $f = (f(x); x \in \mathbb{R}^2)$  be a stationary isotropic centred Gaussian field on  $\mathbb{R}^2$  with constant variance equal to 1,  $\lambda \in \mathbb{R}$ , and  $F = \{x : f(x) \leq \lambda\}$ , and let  $W \in \mathcal{W}$  bounded. Assume that  $f$  has almost surely  $\mathcal{C}^{1,1}$  trajectories and that*

$$\mathbf{ELip}(\partial_1 f, W)^{4+\eta} < \infty$$

for some  $\eta > 0$ . Define  $\mu = \mathbf{E}\partial_1 f(0)^2$ , and  $\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\lambda \exp(-t^2/2) dt$ . Then

$$\mathbf{E}\text{Vol}(F \cap W) = \text{Vol}(W)\Phi(\lambda), \quad (14)$$

$$\mathbf{E}\text{Per}_\infty(F \cap W) = \text{Vol}(W)2\frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2) + \text{Per}_\infty(W)\Phi(\lambda), \quad (15)$$

$$\mathbf{E}\chi(F \cap W) = \left( -\text{Vol}(W)\frac{\mu\lambda}{(2\pi)^{3/2}} + \text{Per}_\infty(W)\frac{\sqrt{\mu}}{4\pi} \right) e^{-\lambda^2/2} + \frac{1}{\sqrt{2\pi}}\Phi(\lambda)\chi(W). \quad (16)$$

**Remark 14.** If  $W$  is a square, the relation 16 coincides with [6, (11.7.14)], obtained under stronger requirements for  $f$ .

*Proof.* (10) immediately yields (14). Let us compute the volumic perimeter  $\text{Per}_\infty(F)$ . To take advantage of the stationarity and isotropy, we study the reduced form of the covariance  $\sigma(r) := \mathbf{E}f(0)f(r\mathbf{u}_i)$ ,  $r > 0$ , which does not depend on  $i$  by isotropy. We have, provided  $\mathbf{E}$  and  $\lim_\varepsilon$  can be switched,

$$\mu = \mathbf{E}\partial_1 f(0)^2 = \mathbf{E} \left( \lim_{\varepsilon \rightarrow 0} \frac{f(\varepsilon\mathbf{u}_1) - f(0)}{\varepsilon} \right)^2 = 2 \lim_{\varepsilon} \frac{1 - \sigma(\varepsilon)}{\varepsilon^2}.$$

This is so if we have a finite bound on

$$\mathbf{E} \sup_{0 < \varepsilon \leq 1} \left( \frac{f(0) - f(\varepsilon\mathbf{u})}{\varepsilon} \right)^2 \leq \mathbf{E} \sup_W |\partial_1 f|^2,$$

which is indeed finite. Put differently,  $\sigma(\varepsilon) = 1 - \frac{1}{2}\mu_\varepsilon\varepsilon^2$  where  $\mu_\varepsilon \rightarrow \mu$  as  $\varepsilon \rightarrow 0$ , for some  $\mathbf{u} \in \mathcal{S}^1$ . Let us first estimate  $p_\varepsilon := \mathbf{P}(f(0) \leq \lambda, f(\varepsilon\mathbf{u}) > \lambda)$ , for some  $\mathbf{u} \in \mathcal{S}^1, \varepsilon > 0$ . Let

$$M = \begin{pmatrix} 1 & \sigma(\varepsilon) \\ \sigma(\varepsilon) & 1 \end{pmatrix}$$

be the covariance matrix of  $(f(0), f(\varepsilon\mathbf{u}_1))$ . We have  $\det(M) = \mu\varepsilon^2 + o(\varepsilon^2)$  and

$$M^{-1} = \frac{1}{\det(M)}(A + \mu_\varepsilon\varepsilon^2 B)$$

where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

and the density kernel of  $(f(0), f(\varepsilon\mathbf{u}_1))$  is

$$\varphi_\varepsilon(t, s) = \frac{1}{2\pi\sqrt{\det(M)}} \exp\left(-\frac{1}{2}(t, s)M^{-1}(t, s)'\right), \quad t, s \in \mathbb{R}^2,$$

where here and in all the rest of the proof, the transposed matrix of any matrix  $H$  is denoted  $H'$ .

Let  $t, s \in \mathbb{R}$ . Denote  $\Lambda = (\lambda, \lambda)$ ,  $X = (t, s)$ , and remark that  $\Lambda' A = A \Lambda' = 0$ . This yields

$$\begin{aligned} \varphi_\varepsilon(t + \lambda, s + \lambda) &= \frac{1}{2\pi\sqrt{\det(M)}} \exp\left(-\frac{1}{2\det(M)}(X'AX + (X + \Lambda)'\mu_\varepsilon\varepsilon^2 B(X + \Lambda))\right) \\ &= \frac{1}{2\pi\sqrt{\det(M)}} \exp\left(-\frac{1}{2\det(M)}(X'AX)\right) \exp\left(-\frac{\mu_\varepsilon\varepsilon^2}{2\det(M)}(2\Lambda'BX + X'BX)\right) \\ &\quad \exp\left(-\frac{\mu_\varepsilon\varepsilon^2}{2\det(M)}\lambda^2\right), \end{aligned}$$

therefore,

$$\begin{aligned} p_\varepsilon &= \frac{1}{2\pi\sqrt{\det(M)}} \int_{t \leq \lambda, s > \lambda} \varphi_\varepsilon(t, s) dt ds = \frac{1}{2\pi\sqrt{\det(M)}} \int_{t \leq 0, s > 0} \varphi_\varepsilon(t + \lambda, s + \lambda) dt ds \\ &= \frac{1}{2\pi\sqrt{\det(M)}} \exp(-(1 + o(1))\lambda^2/2) \\ &\quad \int_{t \leq 0, s > 0} \exp\left(-\frac{1}{2\det(M)}X'AX\right) \exp\left(-\frac{\mu_\varepsilon\varepsilon^2}{2\det(M)}(2\Lambda'BX + X'BX)\right) dX \\ &= \frac{\exp(-\lambda^2/2)\sqrt{\det(M)}}{\pi} (I + R_\varepsilon)(1 + o(1)), \end{aligned}$$

where

$$\begin{aligned} I &= \int_{t \leq 0, s > 0} \exp(-(t - s)^2) dt ds \\ R_\varepsilon &= \int_{t \leq 0, s > 0} \exp(-X'AX) \left[ \exp\left(-\underbrace{\mu_\varepsilon\varepsilon^2 \left[ \frac{\sqrt{2}\Lambda'BX}{\sqrt{\det(M)}} + X'BX \right]}_{\delta_\varepsilon}\right) - 1 \right] dX. \end{aligned}$$

The integrand in  $R_\varepsilon$  goes point-wise to 0 as  $\varepsilon \rightarrow 0$ . For  $t \leq 0, s > 0$ ,  $(t - s)^2 = t^2 + s^2 + 2|ts| \geq t^2 + s^2$ . For  $\varepsilon$  sufficiently small,  $t, s \in \mathbb{R}^2$ ,

$$\delta_\varepsilon \leq \frac{1}{2}(t^2 + s^2).$$

Therefore, the integrand in  $R_\varepsilon$  is uniformly bounded by

$$\exp(-(t - s)^2) \frac{t^2 + s^2}{2} \exp\left(\frac{t^2 + s^2}{2}\right),$$

which is integrable, whence  $R_\varepsilon \rightarrow 0$ . The change of variables  $u = t - s, v = t + s$ , with Jacobian 2, gives

$$I = \frac{1}{2} \int_{u+v < 0, v-u > 0} \exp(-u^2) du dv = \frac{1}{2} \int_{u > 0} 2u \exp(-u^2) du = \frac{1}{2}.$$

Therefore  $\varepsilon^{-1}p_\varepsilon \rightarrow \frac{\varepsilon\sqrt{\mu}}{2\pi} \exp(-\lambda^2/2)$  as  $\varepsilon \rightarrow 0$ , whence

$$\overline{\text{Per}}_{\mathbf{u}_i}(F) = \frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2), \quad \overline{\text{Per}}_\infty(F) = 2 \frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2). \quad (17)$$

Using (9), we obtain (15). In the isotropic case, (8) becomes

$$\mathbf{E}\chi(F \cap W) = \text{Vol}(W)\overline{\chi}(F) + \frac{1}{2}\text{Per}_\infty(W) \lim_{\varepsilon} \varepsilon^{-1}p_\varepsilon + \chi(W)\overline{\text{Vol}}(F).$$

To prove (16), first remark that the stationarity of the field and the fact that it is not constant a.s. entail that  $(f(0), \partial_1 f(0), \partial_2 f(0)) \stackrel{(d)}{=} (f(x), \partial_1 f(x), \partial_2 f(x))$ ,  $x \in \mathbb{R}^2$  is non-degenerated. It therefore remains to show

$$\overline{\chi}(F) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathbf{E} [\delta^\varepsilon(0, f, \lambda) - \delta^{-\varepsilon}(0, -f, -\lambda)] = -\frac{\mu\lambda \exp(-\lambda^2/2)}{(2\pi)^{3/2}}. \quad (18)$$

Fix  $\varepsilon > 0$ . Let  $M$  be the  $3 \times 3$  covariance matrix of  $(f(0), f(\varepsilon\mathbf{u}_1), f(\varepsilon\mathbf{u}_2))$ . Then  $M = U - \frac{1}{2}\mu\varepsilon^2 V_\varepsilon$  where  $V_\varepsilon = V + W_\varepsilon$  with  $W_\varepsilon \rightarrow 0$  and

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

One can then verify that, with

$$V^* := \begin{pmatrix} -4 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}, \quad D := \text{com}(V) = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix},$$

we have  $\det(M) = \varepsilon^4 \frac{\mu^2}{8} \text{Tr}(VV^*) + o(\varepsilon^4) = \varepsilon^4 \mu^2 + o(\varepsilon^4)$ , and for  $\varepsilon$  sufficiently small,

$$M^{-1} = \frac{1}{2\det(M)} \left( -\mu\varepsilon^2(V^* + W_\varepsilon^*) + \frac{\varepsilon^4}{2}\mu^2 D_\varepsilon \right), \quad (19)$$

where  $D_\varepsilon \rightarrow D$ , and  $V^*$  and  $W_\varepsilon^*$  are two matrices for which the sum of each line or of each column is 0, and  $W_\varepsilon^* \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . In particular, noting  $\Lambda = (\lambda, \lambda, \lambda)$ ,  $V^*\Lambda = W_\varepsilon^*\Lambda = 0$ ,  $\Lambda V^* = \Lambda W_\varepsilon^* = 0$  (see the first arXiv version for the interpretation of the  $*$  notation). Also, all matrices involved are symmetric. We have by isotropy and symmetry, for  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} \mathbf{E}\delta^\varepsilon(0, f, \lambda) &= \mathbf{E}\delta^{-\varepsilon}(0, f, \lambda) = \mathbf{E}\delta^\varepsilon(0, -f, \lambda) = \mathbf{P}(f(0) \leq \lambda, f(\varepsilon\mathbf{u}) > \lambda, f(\varepsilon\mathbf{v}) > \lambda) \\ &= \frac{1}{\sqrt{(2\pi)^3 \det(M)}} \int_{\mathbb{R}^3} \mathbf{1}_{\{t \leq \lambda, s > \lambda, z > \lambda\}} \exp\left(-\frac{1}{2}(t, s, z)' M^{-1}(t, s, z)\right) dt ds dz. \end{aligned}$$

Therefore, (8) yields that

$$\overline{\chi}(F) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} (\mathbf{E}\delta^\varepsilon(0, f, \lambda) - \mathbf{E}\delta^\varepsilon(0, f, -\lambda)).$$

Note  $Q = \{(t, s, z) : t \leq 0, s > 0, z > 0\}$ . Let  $t, s, z \in \mathbb{R}$ , put  $X = (t, s, z)$ ,  $Y = \sqrt{\frac{\mu\varepsilon^2}{\det(M)}}X$ . Recalling that  $V^*\Lambda = W_\varepsilon^*\Lambda = 0$ ,  $\Lambda V^* = \Lambda W_\varepsilon^* = 0$ , we have

$$\begin{aligned} (X + \Lambda)'M^{-1}(X + \Lambda) &= \frac{1}{2\det(M)} \left[ -\mu\varepsilon^2(X'(V^* + W_\varepsilon^*)X) + \mu^2\frac{\varepsilon^4}{2}X'D_\varepsilon X + \mu^2\varepsilon^4\Lambda'D_\varepsilon X + \frac{\mu^2\varepsilon^4}{2}\Lambda'D_\varepsilon\Lambda \right] \\ &= \frac{\mu^2\varepsilon^4}{4\det(M)}\Lambda'D_\varepsilon\Lambda - \frac{1}{2}Y'(V^* + W_\varepsilon^*)Y + \frac{\mu\varepsilon^2}{4}Y'D_\varepsilon Y + \frac{\mu^2\varepsilon^4}{2\det(M)}\sqrt{\frac{\det(M)}{\mu\varepsilon^2}}\Lambda'D_\varepsilon Y. \end{aligned}$$

Therefore, with  $\alpha_\varepsilon = \frac{\mu^{3/2}\varepsilon^2}{4\sqrt{\det(M)}} \rightarrow \frac{\sqrt{\mu}}{4}$ ,

$$\begin{aligned} \mathbf{E}\delta^\varepsilon(0, f, \lambda) &= \frac{1}{\sqrt{(2\pi)^3 \det(M)}} \int_Q \exp\left(-\frac{1}{2}(X + \Lambda)'M^{-1}(X + \Lambda)\right) dX \\ &= \left(\sqrt{\frac{\det(M)}{\mu\varepsilon^2}}\right)^3 \frac{1}{\sqrt{(2\pi)^3 \det(M)}} \exp(-(1 + o(1))\lambda^2/2) \\ &\quad \int_Q \exp\left(\frac{1}{4}Y'(V^* + W_\varepsilon^*)Y\right) \exp\left(-\frac{\mu\varepsilon^2}{8}Y'D_\varepsilon Y\right) \exp(-\alpha_\varepsilon\varepsilon\Lambda'D_\varepsilon Y) dY \end{aligned}$$

and, for some  $\theta = \theta(\varepsilon, Y) \in [-1, 1]$ ,

$$\begin{aligned} \mathbf{E}\delta^\varepsilon(0, f, \lambda) - \mathbf{E}\delta^\varepsilon(0, f, -\lambda) &= \frac{\exp(-\lambda^2/2)\det(M)(1 + o(1))}{(2\pi)^{3/2}\mu^{3/2}\varepsilon^3} \\ &\quad \int_Q \exp\left(\frac{1}{4}Y'V^*Y\right) \exp\left(-\frac{\mu\varepsilon^2}{8}Y'D_\varepsilon Y\right) (-2\alpha_\varepsilon\varepsilon\Lambda'D_\varepsilon Y) \exp(-\alpha_\varepsilon\varepsilon\theta\Lambda'D_\varepsilon Y) \exp\left(\frac{1}{4}Y'W_\varepsilon Y\right) dY \\ &= -\frac{\mu\varepsilon^2 \exp(-\lambda^2/2)(1 + o(1))}{2(2\pi)^{3/2}} (I + R_\varepsilon) \end{aligned} \tag{20}$$

where, recalling  $W_\varepsilon \rightarrow 0$ ,  $D_\varepsilon \rightarrow D$ ,  $I = \int_Q \exp(\frac{1}{4}Y'V^*Y)\Lambda'DY dY$  and

$$|R_\varepsilon| \leq \nu \int_Q \exp\left(\frac{1}{4}Y'V^*Y\right) \exp(\nu(\varepsilon + \|W_\varepsilon\|_\infty)\|Y\|^2) \nu\varepsilon|\Lambda'DY|^2 dY$$

for some  $\nu > 0$ . The integrand of  $R_\varepsilon$  decreases point-wise to 0. The matrix  $-V^*$  is positive definite, whence for  $\varepsilon$  sufficiently small,  $R_\varepsilon < \infty$ . It follows by the monotone convergence theorem that  $R_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We have  $I = \lambda(\sqrt{2})^4 2J$  where

$$\begin{aligned} J &= \int_Q \exp(-(2t^2 + s^2 + z^2 - 2ts - 2tz))(s + z) dt ds dz \\ &= \int_Q \exp(-(t - s)^2 - (t - z)^2)(s + z) dt ds dz. \end{aligned}$$

The change of variables

$$\begin{cases} u &= t - s \\ v &= t - z \\ w &= t \end{cases} \Leftrightarrow \begin{cases} t &= w \\ s &= w - u \\ z &= w - v \end{cases}$$

with Jacobian 1 yields  $J = 2J_1 - 2J_2$  where

$$\begin{aligned} J_1 &= 2 \int_{u < v < 0} \exp(-u^2 - v^2) \int_{v < w < 0} w dw dudv \\ J_2 &= \int_{u, v < 0} \exp(-u^2 - v^2) u \int_{\max(u, v)}^0 dw dudv \\ &= - \int_{u < v < 0} \exp(-u^2 - v^2) uv dudv - \int_{v < u < 0} \exp(-u^2 - v^2) u^2 dudv \\ &= - \int_{v < 0} v \exp(-v^2) \left(-\frac{1}{2}\right) [\exp(-u^2)]_{-\infty}^v + J_1 \\ &= -\frac{1}{8} + J_1, \end{aligned}$$

which finally yields  $J = \frac{1}{4}$ . Reporting in (21) entails

$$\bar{\chi}(F) = -\frac{\mu\lambda \exp(-\lambda^2/2)}{(2\pi)^{3/2}}$$

which indeed proves (18).  $\square$

## References

- [1] R. Adler. *The Geometry of Random fields*. John Wiley & sons, 1981.
- [2] R. J. Adler, O. Bobrowski, M. S. Borman, E. Subag, and S. Weinberger. Persistent homology for random fields and complexes. *IMS Coll.*, 6:124–143, 2010.
- [3] R. J. Adler and G. Samorodnitsky. Climbing down Gaussian peaks. [preprint arXiv 1510.07151](#), 2015.
- [4] R. J. Adler, G. Samorodnitsky, and J. E. Taylor. High level excursion set geometry for non-gaussian infinitely divisible random fields. *Ann. Prob.*, 41(1):134–169, 2013.
- [5] R. J. Adler and J. E. Taylor. Euler characteristics for Gaussian fields on manifolds. *Ann. Prob.*, 31(2):533–563, 2003.
- [6] R. J. Adler and J. E. Taylor. *Random Fields and Geometry*. Springer, 2007.
- [7] C. H. Arns, J. Mecke, K. Mecke, and D. Stoyan. Second-order analysis by variograms for curvature measures of two-phase structures. *The European Physical Journal B*, 47:397–409, 2005.
- [8] A. Auffinger and G. Ben Arous. Complexity of random smooth functions on the high-dimensional sphere. *Ann. Prob.*, 41(6):4214–4247, 2013.

- [9] J. Azaïs and M. Wschebor. A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail. *Stoc. Proc. Appl.*, 118(7):1190–1218, 2008.
- [10] H. Biermé and A. Desolneux. On the perimeter of excursion sets of shot noise random fields. to appear in *Ann. Prob.*, 2015.
- [11] P. Cannarsa and C. Sinestrari. *Semi-concave functions, Hamilton-Jacobi equations and Optimal Control*. Birkhäuser, Basel, 2004.
- [12] J. Cao and K. Worsley. The geometry of correlation fields with an application to functional connectivity of the brain. *Ann. Appl. Prob.*, 9:1021–1057, 1999.
- [13] J. Cao and K. J. Worsley. The detection of local shape changes via the geometry of Hotelling’s  $t^2$  fields. *Ann. Stat.*, 27:925–942, 1999.
- [14] A. Estrade and J. R. Leon. A central limit theorem for the Euler characteristic of a Gaussian excursion set. *Ann. Prob.*, to appear, 2014.
- [15] H. Federer. Curvature measures. *Trans. AMS*, 93(3):418–491, 1959.
- [16] B. Galerne and R. Lachièze-Rey. Random measurable sets and covariogram realisability problems. *Adv. Appl. Prob.*, 47(3), 2015.
- [17] R. Hilfer. Review on scale dependent characterization of the microstructure of porous media. *Transport in Porous Media*, 46(2-3):373–390, 2002.
- [18] J. Hiriart-Urruty, J. Strodjot, and V. H. Nguyen. Generalized Hessian matrix and second-order optimality conditions for problems with  $C^{1,1}$  data. *Appl. Math. Optim.*, 11:43–56, 1984.
- [19] J. M. Kilner and K. J. Friston. Topological inference for EEG and MEG. *Ann. Appl. Stat.*, 4(3):1272–1290, 2010.
- [20] R. Lachièze-Rey. Covariograms and Euler characteristic I. Regular sets. [preprint arXiv 1510.00501](#), 2015.
- [21] C. Lantuéjoul. *Geostatistical Simulation: Models and Algorithms*. Springer, Berlin, 2002.
- [22] D. Marinucci. Fluctuations of the Euler-Poincaré characteristic for random spherical harmonics. [preprint arXiv 1504.01868](#), 2015.
- [23] A. L. Melott. The topology of large-scale structure in the universe. *Physics Reports*, 193(1):1 – 39, 1990.
- [24] I. Molchanov. *Theory of random sets*. Springer-Verlag, London, 2005.
- [25] J. Schmalzing, T. Buchert, A. L. Melott, V. Sahni, B. S. Sathyaprakash, and S. F. Shandarin. Disentangling the cosmic web. I. morphology of isodensity contours. *The Astrophysical Journal*, 526(2):568, 1999.
- [26] C. Scholz, F. Wirner, J. Götz, U. Rüde, G.E. Schröder-Turk, K. Mecke, and C. Bechinger. Permeability of porous materials determined from the Euler characteristic. *Phys. Rev. Lett.*, 109(5), 2012.
- [27] J. E. Taylor and K. J. Worsley. Random fields of multivariate test statistics, with applications to shape analysis. *Ann. Stat.*, 36(1):1–27, 2008.